CH 7: Connections

7.1 Affine Connections

Dfn 7.1. **Affine connection** on M is an $\mathbb{R}$-Bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad , \quad (X,Y) \mapsto \nabla_X Y$$

satisfying $\forall X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$

(i) $\nabla_{fX} Y = f \nabla_X Y$

(ii) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$

$\mathbb{R}$-bilinearity says: $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

Abuse of notation: $\partial_i = \frac{\partial}{\partial x_i}$ and $\nabla_i = \nabla_{\partial_i}$

Connection coefficients: $\nabla_i \partial_j = \Gamma_{ij}{}^k \partial_k$ using Einstein summation convention.

Local expression: $\nabla_X Y = X^i \partial_i Y^j \partial_j + X^i Y^j \Gamma_{ij}{}^k \partial_k$

Difference of two affine connections is tensorial:

Lem 7.2: ∇ and ∇' affine connections. The following is a $(1,2)$-tensor field:

$$\kappa : \mathfrak{X}(M) \times \mathfrak{X}(M) \to M \qquad \kappa : (X,Y) \mapsto \nabla_X Y - \nabla'_X Y$$

b.c. κ defines a trilinear map

$$K : \Omega^1(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M) , \quad K(\alpha, X, Y) = \alpha(\kappa(X,Y)).$$

Standard affine connection on $\mathbb{R}^n$: $\Gamma_{ij}{}^k \equiv 0$.

Dfn 7.3: Parallel vector fields: a v.f $Y \in \mathfrak{X}(M)$ such that $\nabla_X Y = 0 \ \forall X \in \mathfrak{X}(M)$ is called parallel.

Lem 7.5: Let ∇' and ∇'' be affine connections on M. Let $\rho_1, \rho_2 \in C^\infty(M)$ with $\rho_1 + \rho_2 = 1$. Then $\nabla = \rho_1 \nabla' + \rho_2 \nabla''$ is an affine connection.

Prop 7.6: $\forall X, Y \in \mathfrak{X}(M)$ define $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $\nabla_X Y := \sum_{i \in I} \rho_i \overset{\alpha(i)}{\nabla}_X Y$.

7.2 : Torsion and Curvature

Dfn 7.1: Torsion of ∇:

$$\left.\begin{aligned} &T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned}\right\} (1,2)\text{-tensor field}$$

prop 7.8

Curvature of ∇:

$$\left.\begin{aligned} &R^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &R^\nabla(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \end{aligned}\right\} (1,3)\text{-tensor field}$$

"flat" = zero curvature.
"torsion-free" = zero torsion.

Torsion coefficients: $T_{ij}{}^k \partial_k = T(\partial_i, \partial_j)$

$\Rightarrow$ $T_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$

Curvature coefficients : $R_{ijk}{}^\ell \partial_\ell = R^\nabla(\partial_i, \partial_j)\partial_k$

$\Rightarrow$ $R_{ijk}{}^\ell = \partial_i \Gamma_{jk}{}^\ell - \partial_j \Gamma_{ik}{}^\ell + \Gamma_{jk}{}^m \Gamma_{im}{}^\ell - \Gamma_{ik}{}^m \Gamma_{jm}{}^\ell$

Dfn 7.2 : Ricci Tensor : $(0,2)$-tensor defined by

$$\mathrm{Ric}^\nabla(X,Y) := \mathrm{tr}(Z \to R(Z,X)Y)$$

$R_{ij} := \mathrm{Ric}(\partial_i, \partial_j) \Rightarrow R_{ij} = R_{kij}{}^k$ } Ric is a contraction of R.

7.3 : Bianchi Identities

Extending by Leibniz rule:

- $f \in C^\infty(M)$, define $\nabla_X(f) := X(f)$
- $\alpha \in \Omega^1(M)$, define $\nabla_X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y)$

Define $\mathfrak{S} A(X,Y,Z) := A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y)$

Prop 7.6: Bianchi identity:

1. **Algebraic Bianchi identity:**

$$\mathfrak{S} R(X,Y)Z = \mathfrak{S}(\nabla_X T)(Y,Z) + \mathfrak{S}\, T(T(X,Y),Z)$$

2. **Differential Bianchi identity:**

$$\mathfrak{S}((\nabla_Z R)(X,Y))W + \mathfrak{S} R(T(X,Y),Z)W = 0$$

7.4 Koszul Connections

Dfn 7.18: A **Koszul connection** on a vector bundle $E \to M$ is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \qquad (X, s) \mapsto \nabla_X s$$

Satisfying $\forall f \in C^\infty(M), X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$:

(i) $\nabla_{fX} s = f \nabla_X s$, and

(ii) $\nabla_X (fs) = X(f)s + f\nabla_X s$

e.g. affine connection = Koszul connection on tangent bundle $TM \to M$.

Let ∇^E and ∇^F be koszul connections on $E \to M$ and $F \to M$ respectively. Induces a koszul connection $\nabla^\otimes$ on the vector bundle $E \otimes F \to M$ as follows:

$$\nabla_X^{E\otimes F}(s \otimes t) := (\nabla_X^E s) \otimes t + s \otimes (\nabla_X^F t)$$

and ∇^{Hom} on the vector bundle $Hom(E,F) \to M$ as follows:

$$(\nabla_X^{Hom} \varphi)(s) := \nabla_X^F(\varphi(s)) - \varphi(\nabla_X^E s).$$

Curvature of koszul connection: $R^\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$

Connection coefficients (for rank-m v.b) are mn^2 functions $\omega_i{}^b{}_a \in C^\infty(U)$ defined by

$$\nabla_i e_a := \omega_i{}^b{}_a e_b$$

where e_a = local sections linearly independent on U.

Local expression of $s = s^a e_a \Rightarrow \nabla_i s = \partial_i s^a e_a + s^a \omega_i{}^b{}_a e_b$

$$= (\partial_i s^a + \omega_i{}^a{}_b s^b) e_a$$

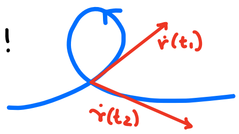
7.5: Parallel Transport:

Smooth curve: a smooth map $\gamma : [0,1] \to M$

Section along γ: smooth map $\sigma : [0,1] \to E$ such that $\pi \circ \sigma = \gamma$, equivalently $\sigma(t) \in E_{\gamma(t)}$.

↳ if $\sigma \in \Gamma(E)$, then $\sigma = s \circ \gamma$ is a section along γ for extendible sections.

NB: not all sections are extendible!!



$\Gamma_\gamma(E)$ = space of sections along γ = $C^\infty([0,1])$-module : $(f\sigma)(t) = f(t)\sigma(t) \in E_{\gamma(t)}$.

any section $\sigma \in \Gamma_\gamma(E)$ can be written locally as $\sigma = \sigma^a(e_a \circ \gamma)$.

Dfn 7.24: ∇ Koszul on $E \to M$. The covariant derivative on $\Gamma_\gamma(E)$ induced by ∇ is the $\mathbb{R}$-linear map

$$\frac{D}{dt} : \Gamma_\gamma(E) \to \Gamma_\gamma(E)$$

characterized by the properties that:

(i) $\forall\ s \in \Gamma(E)$, $\frac{D}{dt}(s \circ \gamma) = (\nabla_{\dot{\gamma}} s) \circ \gamma$

where $\dot{\gamma} = \gamma_*\left(\frac{d}{dt}\right)$

(ii) $\forall f \in C^\infty([0,1])$ and $\sigma \in \Gamma_\gamma(E)$ $\frac{D}{dt}(f\sigma) = \frac{df}{dt}\sigma + f\frac{D\sigma}{dt}$.

parallel / covariantly constant $:= \frac{D\sigma}{dt} = 0$. $\Big\}$ parallel iff $\sigma^a \in C^\infty([0,1])$ obeys ODE $\frac{d\sigma^a}{dt} + \dot{\gamma}^i(t)\,\omega_i{}^a{}_b(\gamma(t))\,\sigma^b(t) = 0$.

Parallel transport: along γ : given $v \in E_{\gamma(0)}$, $\exists!\ \sigma \in \Gamma_\gamma(E)$, $\sigma(0) = v$. Evaluation at 1 gives linear map

$$\mathbb{P}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)} \qquad v \mapsto \sigma(1).$$

Let $s \in \Gamma(E)$ be s.t $\nabla_X s = 0$ $\forall X \in \mathfrak{X}(M)$. Given any curve $\gamma : [0,1] \to M$, $s \circ \gamma$ is a parallel section along γ.

Prop 7.26: $s_1, s_2 \in \Gamma(E)$ s.t $\nabla_X s_1 = \nabla_X s_2 = 0$ $\forall X \in \mathfrak{X}(M)$ and $s_1(p) = s_2(p)$ for some $p \in M$. Then $s_1 = s_2$.

7.6 : Geodesics

Dfn 7.29: ∇ an affine connection on M. A smooth curve $\gamma: I \to M$ with $I \subset \mathbb{R}$ a nontrivial interval is called a geodesic for ∇ if its velocity $\dot{\gamma} := \gamma_*\left(\frac{d}{dt}\right)$, a vector field along γ, is "self-parallel":

$$\frac{D}{dt}\dot{\gamma} = 0.$$

γ is a geodesic iff $\gamma^i(t)$ (where $\dot{\gamma} = \dot{\gamma}^i \partial_i$) satisfy the ODE:

$$\ddot{\gamma}^k(t) + \Gamma_{ij}{}^k(\gamma(t))\,\dot{\gamma}^i(t)\,\dot{\gamma}^j(t) = 0.$$

notice that torsion does not contribute to eqn since second term is symmetric in $i \leftrightarrow j$.

Prop 7.31: $p \in M$ and $v \in T_pM$. $\exists\ \varepsilon > 0$ and $\exists!$ geodesic $\gamma: [0,\varepsilon) \longrightarrow M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

notation: $\gamma_{p,v}$:= unique geodesic with ics $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

Cor 7.32: $\forall\ p \in M$, $v \in T_pM$, and $s > 0$, $\exists\ \varepsilon > 0$ such that $\gamma_{p,sv}(t) = \gamma_{p,v}(st)$ for $t \in [0,\varepsilon)$.

Dfn 7.33: (Exponential map) ∇ affine connection on M and $p \in M$. Exponential map $\exp_p: T_pM \to M$ is defined to be $\exp_p(v) = \gamma_{p,v}(1)$.

Prop 7.34: In normal coordinates at $p \in M$, the connection coefficients satisfy

$$\Gamma_{ij}{}^k(p) + \Gamma_{ji}{}^k(p) = 0$$

Prop 7.35: $\gamma: I \to M$ a geodesic of affine connection ∇. Let $\phi: M \to M$ be a diffeomorphism. Then $\gamma^\phi: I \to M$, defined by $\gamma^\phi = \phi \circ \gamma$, is a geodesic of the transformed connection ∇^ϕ.

$\Rightarrow$ Geodesic can also be described by the general equation

$$\frac{D}{dt}\dot{\gamma} = g(t)\dot{\gamma}$$

for some function g. Affine parameter t for a geodesic is one relative to which the velocity is self-parallel, so that $g(t) = 0$. Two affine parameters s and t are then related by $t = as + b$ for $a > 0$, $b \in \mathbb{R}$.

CH 8 Riemannian Geometry

8.1 The Metric Tensor

8.1.1 Inner Products

Inner product: a non-degenerate symmetric bilinear form $\langle -,- \rangle : V \times V \to \mathbb{R}$

$\langle v, w \rangle = \langle w, v \rangle$ $\quad \langle v_1 + \alpha v_2, w \rangle = \langle v_1, w \rangle + \alpha \langle v_2, w \rangle$

Non-degenerate: $\langle v, w \rangle = 0 \ \forall w \in V$, then $v = 0$.

Inner product space = real vector space with an inner product

Isometry := isomorphism $\phi : V \to W$ between inner product spaces such that $\forall v_1, v_2 \in V$,

$$\langle \phi(v_1), \phi(v_2) \rangle_W = \langle v_1, v_2 \rangle_V$$

$\mathbb{E}^{s,t}$:= $(s+t = n)$ is the vector space $\mathbb{R}^n$ with the inner product

$$\langle (v_1, \dots, v_n), (w_1, \dots, w_n) \rangle = \sum_{i=1}^{s} v_i w_i - \sum_{j=s+1}^{n} v_j w_j$$

Orthonormal: basis $(e_1, \dots, e_n)$ satisfying $\langle e_i, e_j \rangle = \pm \delta_{ij}$.

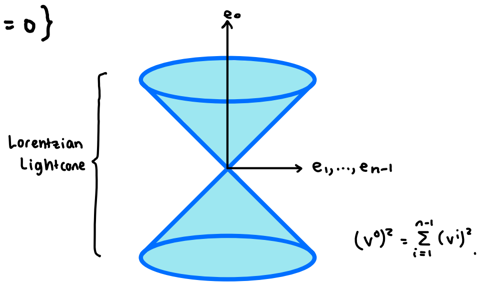
Lem 8.2: Every n-dimensional inner product space $(V, \langle -,- \rangle)$ is isometric to $\mathbb{E}^{s,t}$ for some s and t with $s+t=n$.

Sylvester's Law of Inertia: Signature (s,t) is uniquely determined by inner product.

Euclidean vector space: $(n, 0)$, $\eta = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

Lorentzian vector space: $(n-1, 1)$, $\eta = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

Light cone: $\mathbb{L} = \{ v \in V \mid \langle v, v \rangle = 0 \}$



Nonzero vector is said to be:

Spacelike: $\langle v, v \rangle > 0$

Lightlike/null: $\langle v, v \rangle = 0$

Timelike: $\langle v, v \rangle < 0$

$$\begin{cases} \textit{future} = \text{timelike and } v^0 > 0 \\ \textit{past} = \text{timelike and } v^0 < 0 \\ \vdots \ \text{ spacelike separated from the origin (spacelike vectors).} \end{cases}$$

Notation: Symmetric product $uv := \frac{1}{2}(u \otimes v + v \otimes u)$.

8.1.2 Metrics

Dfn 8.3 (Metric): A metric of signature (s,t) with $s+t=n$ on an n-dimensional manifold M is a section $g \in \Gamma(\odot^2 T^*M)$ such that for all $s \in M$, g_p defines an inner product on T_pM of signature (s,t).

Local expression: $g = g_{ij}\, dx^i dx^j$ where $g_{ij} = g(\partial_i, \partial_j) \in C^\infty(U)$.
$\forall p \in U$, $[g_{ij}(p)]$ is invertible.

$\boxed{g(a,b) = a^T [g] b}$

Remannian Manifold: (M,g) = manifold equipped with a metric.

Lorentzian Manifold: (M,g) with signature $(n-1,1)$.

Dfn 8.7: (M,g), (N,h) be two Riemannian manifolds. A diffeomorphism $F: M \to N$ is called an isometry if $F^*h = g$; that is, if $\forall p \in M$ and $X_p, Y_p \in T_pM$,

$$g_p(X_p, Y_p) = h_{F(p)}((F_*)_p X_p, (F_*)_p Y_p).$$

Local isometry at $a \in M$ if there's a neighbourhood $U \subset M$ of a s.t $F|_U : U \to F(U)$ is an isometry.
↳ notion of (M,g) and (N,h) being locally isometric.

Isometries form a subgroup of the group of diffeomorphisms

Dfn 8.8: A vector field $X \in \mathfrak{X}(M)$ on a Riemannian manifold is called a "Killing vector field" if $\mathcal{L}_X g = 0$. That is, if for all $Y, Z \in \mathfrak{X}(M)$,

$$X g(Y,Z) = g([X,Y],Z) + g(Y,[X,Z]).$$

note: Lie Bracket of two Killing vectors is a Killing vector

(M,g) a Riemannian manifold and $F: N \to M$ an embedding. The pullback F^*g defines a metric on N
↳ An embedding $F: N \to M$ is called isometric if $h = F^*g$.

Example 8.9: round metric on S^2

Example 8.11: Minkowski spacetime, $M = \mathbb{R}^4$, (t,x,y,z) and metric $\eta = -dt^2 + dx^2 + dy^2 + dz^2$.

8.2 The Levi-Civita Connection

Thm 8.12 (Fundamental Theorem of Riemannian Geometry): For a Riemmanian manifold (M,g), $\exists!$ torsion-free affine connection ∇ which is compatible with the metric: $\nabla g = 0$.

Koszul formula for ∇: $2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g(Z,[X,Y]) - g(Y,[X,Z]) - g(X,[Y,Z])$.

↳ ∇ is known as the "Levi-Civita" connection.

Christoffel symbols: connection coefficients relative to a local chart of the Levi-Civita connection.

: $\Gamma_{ij}{}^m = \frac{1}{2} g^{km}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

(notice $\Gamma_{ij}{}^m = \Gamma_{ji}{}^m$).

⇒ Christoffel symbols vanish in a local chart if the metric coefficients are constant in that chart.

Prop 8.16: Let X be a Killing vector ($\mathcal{L}_X g = 0$). Then X obeys the Killing equation:
$\forall\, Y, Z \in \mathfrak{X}(M)$,

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

Prop 8.17: $\forall\, U, V \in \Gamma_\gamma(TM)$ along γ

$$\frac{d}{dt} g(U,V) = g\left(\frac{DU}{dt}, V\right) + g\left(U, \frac{DV}{dt}\right).$$

Cor 8.18: $\gamma : I \to M$ a geodesic for the Levi-Civita connection of a Riemannian manifold (M,g).
Then $g(\dot\gamma, \dot\gamma)$ is constant along γ.

$$\left.\begin{array}{ll} \textit{Timelike:} & g(\dot\gamma,\dot\gamma) < 0 \\ \textit{Lightlike or null:} & g(\dot\gamma,\dot\gamma) = 0 \end{array}\right\} \le 0 = \textit{causal}$$
Spacelike: $g(\dot\gamma,\dot\gamma) > 0$

Ex. 8.19: Minkowski spacetime: geodesic eqn has linear components relative to an affine parameter.

$\phi : M \to M$ an isometry, and $M := \{p \in M \mid \phi(p) = p\}$. Every connected component of M^ϕ is a closed submanifold of M.

Totally geodesic: A submanifold such that geodesics which start tangent to the submanifold remain there.

Prop 8.22: (M,g) and $\phi : M \to M$ an isometry. Then the fixed point submanifold M^ϕ is totally geodesic.

8.3 The Riemann Curvature Tensor

Riemman curvature Tensor: $(0,4)$-tensor $\left(\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^{\infty}(M) \right)$ defined by

$$\mathrm{Riem}(X,Y,Z,W) := g(R(X,Y)Z,W)$$

Prop 8.23: $\forall\, X,Y,Z,W \in \mathfrak{X}(M)$:

- $\mathrm{Riem}(X,Y,Z,W) = -\mathrm{Riem}(Y,X,Z,W)$
- $\mathrm{Riem}(X,Y,Z,W) + \mathrm{Riem}(Y,Z,X,W) + \mathrm{Riem}(Z,X,Y,W) = 0$
- $\mathrm{Riem}(X,Y,Z,W) = -\mathrm{Riem}(X,Y,W,Z)$
- $\mathrm{Riem}(X,Y,Z,W) = R(Z,W,X,Y)$

Local coords: Riem is determined by its coefficients $R_{ijk\ell} := \mathrm{Riem}(\partial_i, \partial_j, \partial_k, \partial_\ell)$, given by

$$R_{ijk\ell} = g(R(\partial_i,\partial_j)\partial_k, \partial_\ell) = (R_{ijk}{}^{m}\partial_m, \partial_\ell) = R_{ijk}{}^{m} g_{m\ell}.$$

Have identities: $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$ and $R_{[ijk]\ell} = 0$.

Ricci tensor: $R_{ij} = R_{ji}$ for Levi-Civita connection

Ricci scalar: $R := g^{ij} R_{ij}$

Dfn 8.27: Einstein tensor of Levi-Civita connection is a symmetric $(0,2)$-tensor Ein defined by:

$$\mathrm{Ein}(X,Y) := \mathrm{Ric}(X,Y) - \tfrac{1}{2} g(X,Y) R$$

Local coords: determined by it's coefficients: $G_{ij} := R_{ij} - \frac{1}{2} g_{ij} R$

Prop 8.28: $\nabla^j G_{jk} := g^{ij} \nabla_i G_{jk} = 0$ "zero divergence"

Dfn 8.29: (M,g) is said to be Einstein if the Ricci tensor is proportional to the metric:

$$\mathrm{Ric} = \lambda g \quad (\text{or equiv}) \quad R_{ij} = \lambda g_{ij}.$$

for some $\lambda \in \mathbb{R}$. Ricci-flat $:= \lambda = 0$.

8.4 Cartan's method of moving frames

Dfn 8.32 (M^n, g) a Riemannian manifold and $U \subset M$ an open set. A local orthonormal frame for M on U is a collection $(X_1, \dots, X_n)$ with $X_i \in \mathfrak{X}(U)$ such that $g(X_i, X_j) = \pm\delta_{ij}$ everywhere on U.

$\Longrightarrow$ (X_i) are linearly independent on U and $\forall p \in U$, $((X_1)_p, \dots, (X_n)_p)$ is an orthonormal frame for the inner product space (T_pM, g_p).

Prop 8.33: (M,g) and $p \in M$. Then $\exists$ local chart (U, x^i) with $p \in U$ and a local orthonormal frame on U.

(M,g) has signature (s,t) : reorder frame so that $\forall p \in U$, $(e_1, \dots, e_n)$ gives an isomorphism $T_pM \cong \mathbb{E}^{s,t}$

notation: $\eta_{ab} := g(e_a, e_b)$

$(\theta^1, \dots, \theta^n)$ = local coframe canonically dual to $(e_1, \dots, e_n)$ on U.

↳ $\theta^a \in \Omega^1(U)$, $\theta^a(e_b) = \delta^a_b$.

↳ then g has local expression $g = \eta_{ab}\,\theta^a\theta^b$.

Ex 8.34: $e_a^i \theta_i^b = \delta_a^b$, $e_a^i \theta_j^a = \delta^i_j$ for $e_a = e^i_a \partial_i$ and $\theta^a = \theta^a_i dx^i$.

Connection one-forms: $\omega^b{}_a \in \Omega^1(U)$: $\nabla_X e_a = \omega(X)^b_a\, e_b$

$\omega_{ab} := \eta_{ac}\,\omega^c{}_b$: $\omega_{ab} = -\omega_{ba}$

Prop 8.35: First structure Equation : $d\theta^a + \omega^a{}_b \wedge \theta^b = 0$

Prop 8.36: Second structure Equation : $d\omega^a{}_b + \omega^a_c \wedge \omega^c_b = \Omega^a{}_b$,

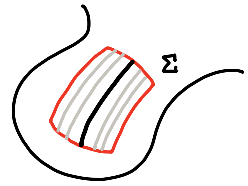
where the curvature two-form $\Omega^a{}_b$ is defined by $\Omega(X,Y)^a{}_b\, e_a = R(X,Y)\, e_b$

↳ $\Omega_{ab} := \Omega^c{}_b\,\eta_{ac} \rightarrow \Omega_{ab} = -\Omega_{ba}$

Prop. 8.38. $R_{ijk\ell} = -\Omega_{ij\,ab}\,\theta_k^a\theta_\ell^b$ where $\Omega_{ij\,ab} = \Omega(\partial_i, \partial_j)_{ab}$.

Important identity: $\frac{1}{2}\Omega_{ab}(\theta^a \wedge \theta^b) = -\frac{1}{4} R_{ijk\ell}\,(dx^i \wedge dx^j)(dx^k \wedge dx^\ell)$ $\Big\}$ denoting symmetric product of two-forms!

8.5 Geodesic Deviation



o.p.f. of geodesics: a smooth map

$$\gamma: (-\varepsilon, \varepsilon) \times [0,1] \to M \quad , \quad (s,t) \mapsto \gamma_s(t),$$

where for every fixed $s \in (-\varepsilon, \varepsilon)$, $\gamma_s : [0,1] \to M$ is an affinely parametrised geodesic.

map parametrizes a 2D surface on M, Σ.

notation: $\dot{\gamma} = \gamma_*\left(\frac{\partial}{\partial t}\right)$, $\gamma' = \gamma_*\left(\frac{\partial}{\partial s}\right)$

Geodesic deviation := acceleration of γ' as we move along the geodesic $\frac{D^2}{dt^2}\gamma'$

Lem 8.42 $\frac{D}{ds}\dot{\gamma} = \frac{D}{dt}\gamma'$

Lem 8.43 : Let X be a vector field along Σ. Then $\frac{D}{dt}\frac{D}{ds}X - \frac{D}{ds}\frac{D}{dt}X = R(\dot{\gamma}, \gamma')X$.

Prop 8.44 (Jacobi Equation / Geodesic deviation equation) : $\frac{D^2}{dt^2}\gamma' = R(\dot{\gamma}, \gamma')\dot{\gamma}$

$\Longrightarrow$ Geodesic deviations determine the curvature tensor.

Lem 8.46: R be the curvature tensor of a torsion free affine connection on a manifold M.
Then $\forall X, Y, Z \in \mathfrak{X}(M)$,

$$R(X,Y)Z = \frac{1}{3}\Big(R(X,Y)Z + R(Z,Y)X - R(Y,X)Z - R(Z,X)Y \Big)$$

Local expression: $R_{ijk}{}^{\ell} = \frac{2}{3}\left(R_{i(jk)}{}^{\ell} - R_{j(ik)}{}^{\ell}\right)$

8.6 The Hodge Star Operator

Let (M^4, g) be lorentzian, $p \in M$. $\exists\, U \subset M$ s.t $p \in U$.
Denote the coframe $\theta^a \in \Omega^1(U)$, $g|_U = \eta_{ab}\,\theta^a\theta^b$, $[\eta_{ab}] = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I \end{array}\right)$

Dfn: volume form: $\exists\, \omega \in \Omega^4(M)$ that is nowhere vanishing, $\omega|_U = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$

Define inner product on $\Omega^1(M)$ by $\langle \theta^a, \theta^b \rangle = \eta^{ab}$, and extend linearly. We can extend this to k-forms by:

$$\langle \theta^a \wedge \theta^b, \theta^c \wedge \theta^d \rangle := \det \begin{pmatrix} \langle \theta^a, \theta^c \rangle & \langle \theta^a, \theta^d \rangle \\ \langle \theta^a, \theta^b \rangle, & \langle \theta^c, \theta^d \rangle \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \langle \theta^0\wedge\theta^1, \theta^0\wedge\theta^1 \rangle = -1 \\ \langle \theta^1\wedge\theta^2, \theta^1\wedge\theta^2 \rangle = 1 \\ \quad \cdots \end{array}$$

Dfn: (Hodge star operator): $* : \Omega^p(M) \to \Omega^{4-p}(M)$, $C^\infty(M)$-linear. e.g. $*: \Omega^2(M) \to \Omega^2(M)$, by

$$\alpha, \beta \in \Omega^2(M), \Rightarrow \alpha \wedge *\beta := \langle \alpha, \beta \rangle \omega$$

Rem: $*(*\alpha) = -\alpha$

Rem: enough to know $*(\theta^a \wedge \theta^b)$

E.g. $(\theta^0 \wedge \theta^1) \wedge *(\theta^0 \wedge \theta^1) = \langle \theta^0\wedge\theta^1, \theta^0\wedge\theta^1 \rangle\, \omega = \langle \theta^0\wedge\theta^1, \theta^0\wedge\theta^1 \rangle$
$= -\,\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$

read off that $*(\theta^0 \wedge \theta^1) = -\theta^2 \wedge \theta^3$

Denote $\theta^{ab} := \theta^a \wedge \theta^b$

α	$*\alpha$
θ^{01}	$-\theta^{23}$
θ^{02}	θ^{13}
θ^{03}	$-\theta^{12}$
θ^{12}	θ^{03}
θ^{13}	$-\theta^{02}$
θ^{23}	θ^{01}

CH 9 Relativity

9.1 Galilean Relativity

Affine space: $\mathbb{A}^n$ (over $\mathbb{R}$) visualise as affine hyperplane in $\mathbb{R}^{n+1}$ (e.g. last entry equal to 1).

Galilean universe: $\mathbb{A}^4$ = space of events with:

Clock: $\tau : \mathbb{R}^4 \to \mathbb{R}$: $\tau(a-b)$ = time between a and b.

↳ **Simultaneous**: $\tau(a-b) = 0$

Ruler: $\Delta : \ker \tau \to \mathbb{R}$: gives Euclidean distance between simultaneous events.

Affine group: $\mathrm{Aff}(n, \mathbb{R}) \subset GL(n+1, \mathbb{R})$ which preserves $\mathbb{A}^n \subset \mathbb{R}^{n+1}$.

↳ $\left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in GL(n, \mathbb{R}), \ b \in \mathbb{R}^n \right\}$

↳ $(x, 1)$, $x \mapsto Ax + b$

Galilean group: subgroup of $\mathrm{Aff}(n, \mathbb{R})$ preserving clock and ruler.

↳ $\left\{ \begin{pmatrix} R & v & a \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} : v, a \in \mathbb{R}^3, \ R \in O(3), \ s \in \mathbb{R} \right\}$

breaks down into:

- **Translation**: $\begin{pmatrix} I & 0 & a \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ t+s \\ 1 \end{pmatrix}$
- **Galilean boost**: $\begin{pmatrix} I & v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} x+tv \\ t \\ 1 \end{pmatrix}$ (v = "velocity")
- **Reorientation of axes**: $\begin{pmatrix} R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx \\ t \\ 1 \end{pmatrix}$

Principal idea of Galilean Relativity: "inertial coordinates" : free particles move in straight lines ("world line of particle")

9.2. Special Relativity

Laplacian: $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

d'alembertian: $\square := \frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

Relative to Minkowski coords, $\square = -\eta_{\mu\nu}\partial_\mu \partial_\nu$

Minkowski Universe: $\mathbb{A}^4$ = Space of spacetime events + "proper distance" between two spacetime events $a,b \in \mathbb{A}^4$:

$$\Delta(a,b) = \eta(b-a,\, b-a)$$

η = minkowski inner product defined on $\mathbb{R}^4$

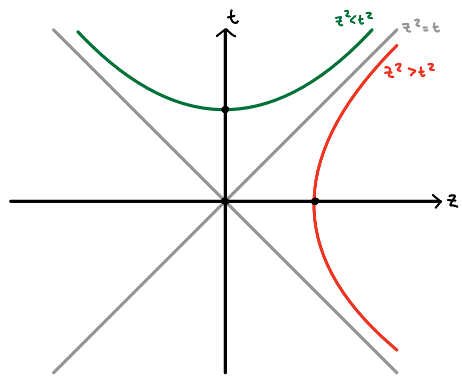
$b-a = (t,x,y,z) \implies \Delta(a,b) = -t^2 + x^2 + y^2 + z^2.$

$-$ve prop. dist =: proper time.

Poincaré group: Subgroup of $\mathrm{Aff}(4,\mathbb{R})$ preserving proper distance.

$\hookrightarrow \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in O(3,1),\ v \in \mathbb{R}^4 \right\}$ $\qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

where **Lorentz group:** $O(3,1) = \{ A \in GL(4,\mathbb{R}) : A^T \eta A = \eta \}$,



Lorentz boost: $(t,x,y,z) \mapsto (t',x,y,z')$,

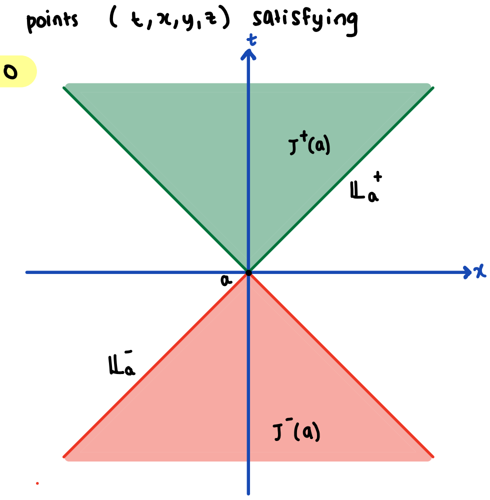
$\downarrow$ where by linearity $z' = \alpha z + \beta t$, $t' = \gamma z + \delta t$.

Reparametrise:
$z' = z\cosh\zeta + ct\sinh\zeta$,
$t' = \frac{1}{c} z \sinh\zeta + t\cosh\zeta$.

$\nexists$ nonzero Poincaré-invariant clock $\tau : \mathbb{R}^4 \to \mathbb{R}$ in Minkowski spacetime.

$\mathbb{L}_a$ = lightcone based at a: if $a = (t_0, x_0, y_0, z_0)$, then $\mathbb{L}_a$ consists of points (t,x,y,z) satisfying

$$-(t-t_0)^2 + (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 0$$



$\mathbb{L}_a^+$ = future lightcone $\quad t > t_0$
$\mathbb{L}_a^-$ = past lightcone $\quad t < t_0$.

a, b are *causally related* if $\Delta(a,b) \leq 0$.
a, b are *spacelike separated* if $\Delta(a,b) > 0$

$J^+(a)$ = causal future: points at a non+ proper dist. to a
$J^-(a)$ = causal past: points at a non- proper dist. to a

Relavistic equations: PDE's for functions on Minkowski spacetime that are invariant under $\square$
Maxwell's equations in vacuo: $dF = 0$ and $d * F = 0$

9.3 General Relativity

Postulates of GR:

1. Spacetime is a connected, 4D Lorentzian manifold (M, g)

2. Free particles in the spacetime follow timelike or null geodesics of the Levi-Civita connection

Minimal coupling: go from SR to GR

3. The distribution of matter (including radiation) is described by the energy-momentum tensor T, a symmetric $(0,2)$-tensor which has zero divergence $\nabla^{\mu} T_{\mu\nu} = 0$.

Exm 9.18: Maxwell's equations: let $F \in \Omega^2(M)$, $F = \frac{1}{2} F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$.
Equations are: $dF = 0$ $\quad g^{\mu\nu} \nabla_{\mu} F_{\nu\rho} = 0$

Energy-momentum tensor of F: $T_{\mu\nu} = g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} g^{\rho\tau} g^{\sigma\chi} F_{\rho\sigma} F_{\tau\chi}$

4. The curvature of the spacetime is related to the energy-momentum tensor via the Einstein Equation

$$\mathrm{Ein} = 8\pi G T$$

or relative to a local chart,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Taking trace: Einstein equation then becomes $\mathrm{Ric} = \Lambda g$, where Λ is called the "cosmological constant".

CH 10 Exact Solutions

10.1 Isometric actions of Lie Groups

10.1.1 Group actions on Riemannian manifolds

$G :=$ Lie group, M a manifold.

Left action of G on M is a smooth map $\alpha : G \times M \to M$, $(g,p) \mapsto g \cdot p$ satisfying:
1. $\forall p \in M$, $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$
2. $\forall p \in M$, $e \cdot p = p$

$\forall g \in G$, get an associated diffeo $\phi_g : M \to M$, $p \mapsto g \cdot p$. Note: $\phi_g \circ \phi_h = \phi_{gh}$

A left action is
- **Effective:** $\ker \phi = \{e\}$
- **Transitive:** $\forall p, q \in M$, $\exists g$ s.t $p = g \cdot q$, equivalently $G \cdot p = M$. $\forall p$

Lem 10.1: G compact. Then $G \cdot p \subset M$ is an embedded subman. of M $\forall p$.

Fundamental vector field: $\mathfrak{g} = T_e G$, and $X \in \mathfrak{g}$. Then X defines a vector field $\tilde{X}$ on M as follows: Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a curve with $\gamma(0) = e$ and $\gamma'(0) = X \in \mathfrak{g}$. Given $p \in M$, let $c : (-\varepsilon, \varepsilon) \to M$ be the curve in M s.t $c(t) = \gamma(-t) \cdot p$. Then $\tilde{X}_p := c'(0) \in T_p M$: $\forall f \in C^\infty(M)$,

$$(\tilde{X} f)(p) = \frac{d}{dt} f(\gamma(-t) \cdot p)\Big|_{t=0}$$

Lem 10.3: $\forall X, Y \in \mathfrak{g}$, $[\tilde{X}, \tilde{Y}] = \widetilde{[X,Y]}$

$G_p :=$ Stabilizer of p

Thm 10.5 (Orbit-Stabilizer Theorem) Let G act smoothly on M and $p \in M$. $\exists$ a G-equivariant diffeo

$$\varphi : G \cdot p \to G / G_p$$

defined by $\varphi(p) = G_p$ and $\varphi(g \cdot p) = g G_p$.
Clearly G-equivariant: $\varphi(g \cdot p) = g \cdot \varphi(p)$.

10.1.2: Homogeneous Riemannian Manifolds

Dfn: (M, g) is **homogeneous** if it admits a transitive isometric action of a Lie Group.
I.e $\exists$ Lie Group G s.t $\forall$ $p \in M$, $G \cdot p = M$ and G preserves the metric.

Ex: Minkowski is hom: translations act transitively.

Thm 10.8 (Frobenius Reciprocity): 1-1 correspondence between G_p-invariant tensors on $T_p M$ and G-invariant tensor fields on M.

10.2 Spherical Symmetry

$G :=$ Lie group $SO(3)$, orbits: $r \in [0, \infty)$

Dfn 10.9: (Spherical symmetry) (M,g) a 4D Lorentzian man. on which G acts by isometries. (M,g) is spherically symmetric if the generic G-orbits are 2-spheres.

Prop 10.14: (M,g) a spherically symmetric spacetime. Then either it is isometric to the Riemannian product $(K, g_K) \times (S^2, r_0 g_{S^2})$ of a Lorentzian two-dimensional manifold and a round sphere of radius r_0, or else $\exists$ chart with local coordinates (τ, r, θ, ϕ), relative to which the metric takes the form

$$g = g_{\tau\tau}(\tau, r)\, d\tau^2 + g_{rr}(\tau, r)\, dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2).$$

10.3 Schwarzschild Metric

Deriving a Ricci-flat metric.

10.3.1 Deriving the Schwarzschild Metric

Dfn 10.15: (M,g) is called ...

Stationary: if it has a timelike $(g(X,X)<0)$ Killing vector, X

Static: stationary and the one form $\theta = X^\flat$ canonically dual to X satisfies $\theta \wedge d\theta = 0$.

↳ X is then "*hypersurface orthogonal*"

E.g. Minkowski, $X = \frac{\partial}{\partial t}$ (clearly $g(X,X) = -1 < 0$).

Ex 10.17: $\theta \in \Omega^1(M)$, $\theta \wedge d\theta = 0$

$\Rightarrow \exists\ \omega \in \Omega^1(M)$ s.t $d\theta = \theta \wedge \omega$,

$\Rightarrow$ $Y, M \in \mathfrak{X}(M)$ s.t $\theta(Y) = \theta(Z) = 0$, $\Rightarrow \theta([Y,Z]) = 0$

Schwarzschild metric: $g = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

$r \to \infty$, $g \to$ Minkowski

↳ "g is asymptotically flat"

10.3.2: Singularities of the Schwarzschild metric.

Singularities: $r = r_s := 2GM$, $r = 0$

Coordinate Singularity (for $r = r_s$); physical Singularity (for $r = 0$)

Kretschmann Scalar: $K := \frac{1}{4} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$

Prop 10.20: The Kretschmann Scalar for the Schwarzschild metric is given by $K = \frac{12M^2}{r^6}$

$\Rightarrow$ $r=0$ is actually a physical singularity.

10.4: The Schwarzschild Black Hole

10.4.1: The Rindler Wedge:

2D Lorentzian metric: $g = dx^2 - x^2 dt^2$, $(x,t) \in (0,\infty) \times \mathbb{R}$

10.4.2: The Kruskal Extension

Concentrate on (r,t) part of Schwarzschild metric: $g = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$

rewriting using change of coordinates: total metric: $g = 16M^2 \frac{1}{r} e^{-r/2M}(dX^2 - dT^2) + r(d\theta^2 + \sin^2\theta\, d\phi^2)$

Dfn 10.24: (M,g) be a spacetime and $j: \Sigma \to M$ an embedded hypersurface. We can pull back g via j onto Σ to give a symmetric $(0,2)$-tensor field $j^*g \in \Gamma(\odot^2 T^*\Sigma)$. We say that Σ is ...

- *spacelike:* j^*g is positive-definite
- *timelike:* j^*g is a lorentzian metric
- *null:* j^*g is degenerate